Discrete Mathematics 73 (1988/89) 27–35
North-Holland

A COMBINATORIAL INTERPRETATION OF $(1/k!)\Delta^k t^n$

A. BENZAIT and B. VOIGT
University of Bielefeld, Postfach 8640, 4800 Bielefeld 1, F. R. Germany

Received 13 January 1988

We consider noncentral Stirling numbers $S_k^n(t) = (1/k!)\Delta^k t^n$ and give a combinatorial interpretation.

1.

Stirling numbers of the second kind may be defined recursively:

$$S_k^{n+1} = S_{k-1}^n + k \cdot S_k^n$$

with boundary conditions $S_k^n = 0$ for $n < 0$ or $k < 0$ or $k > n$ and $S_0^0 = 1$.

In the calculus of finite differences Stirling numbers of the second kind occur as iterated forward differences of the polynomial x^n, evaluated at $x = 0$. The usual notation is

$$S_k^n = \frac{1}{k!} \Delta^k 0^n.$$

An explicit calculation shows that

$$S_k^n = \sum_{a=0}^{k} a^n \cdot \prod_{b=0, b \neq a}^{k} (a-b)^{-1},$$

or, by the Residue Theorem,

$$S_k^n = \frac{1}{2\pi i} \oint \frac{x^n}{\prod_{a=0}^{k} (x-a)} dx.$$

Finally, using divided differences,

$$S_k^n = [0, 1, \ldots, k]x^n.$$

For further explanations the reader should compare any classical textbook on the calculus of finite differences, e.g. [7].

Let us summarize these expressions saying that Stirling numbers of the second kind are derived from interpolating x^n at positions $0, 1, 2, 3, \ldots$.

More generally, the *noncentral* Stirling numbers of the second kind are derived from interpolating x^n at positions $t, t+1, t+2, \ldots$.

Define

$$S_k^n(t) := [t, t+1, \ldots, t+k]x^n$$
$$= \frac{1}{2\pi i} \oint \frac{x^n}{\prod_{a=0}^{k}(x-t-a)}\, dx$$
$$= \sum_{a=0}^{k} (t+a)^n \cdot \prod_{b=0, b\neq a}^{k} (a-b)^{-1}$$
$$= \frac{1}{k!} \Delta^k t^n.$$

One easily verifies that the noncentral Stirling numbers of the second kind can also be derived from the recursion

$$S_k^{n+1}(t) = S_{k-1}^n(t) + (t+k) \cdot S_k^n(t)$$

with the boundary conditions $S_k^n(t) = 0$ for $n < 0$ or $k < 0$ and $S_0^0(t) = 1$.

Remarks. The characterization $S_k^n(t) = (1/k!)\Delta^k t^n$ shows that noncentral Stirling numbers of the second kind occur quite naturally. They have been studied extensively by Carlitz [2] and Koutras [8] where these numbers are denoted $R(n, k, t)$ and $S_{-t}(n, k)$, respectively.

Somewhat more generally Bach [1], Comtet [3] and Voigt [12] consider numbers $S_k^n(a_0, \ldots, a_k) := [a_0, \ldots, a_k]x^n$. Alternatively, these numbers can be defined by the recursion

$$S_k^{n+1}(a_0, \ldots, a_k) = S_{k-1}^n(a_0, \ldots, a_{k-1}) + a_k \cdot S_k^n(a_0, \ldots, a_k)$$

with the obvious boundary conditions. For nonnegative integers a_i [12] gives a combinatorial interpretation for $S_k^n(a_0, \ldots, a_k)$. For $t \geq 0$ a combinatorial interpretation of $S_k^n(t)$ has been also given in [2] and [8]:

Theorem. *For $t \geq 0$ the noncentral Stirling numbers $S_k^n(t)$ count the number of equivalence relations on $\{0, \ldots, t-1, t, \ldots, t+n-1\}$ into precisely $t+k$ mutually disjoint and nonempty equivalence classes such that the elements $0, \ldots, t-1$ belong to mutually distinct classes.*

The main purpose of this paper is to give a combinatorial interpretation of the numbers $S_k^n(t)$ for negative values of t.

In Section 2 we give another interpretation of the numbers $S_k^n(t)$ for $t \geq 0$. In Section 3 we derive some easy observations concerning the case $t \leq 0$. In Section 4 we give the interpretation for $t \leq 0$.

2. Graham–Rothschild parameter words

Let A be a finite set, say, $|A| = t$ and $A \cap \{\lambda_0, \lambda_1, \ldots\} = \emptyset$. By $[A]\binom{n}{k}$ we denote the set of mappings $f : \{0, \ldots, n-1\} \to A \cup \{\lambda_0, \ldots, \lambda_{k-1}\}$ satisfying

(1) $f^{-1}(\lambda_i) \neq \emptyset$ for $i < k$ and

(2) $\min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_j)$ for $i < j < k$.

Such mappings are called *k-parameter words of length n over alphabet A*, representing k-parameter subsets of A^n. Parameter sets have been introduced and studied by Graham and Rothschild [5]. Compare [6], [4] or [10]. Using these definitions we have the following combinatorial interpretation of the numbers $S_k^n(t)$ for $t \geq 0$:

Theorem. $|[A]\binom{n}{k}| = S_k^n(t)$, *where* $|A| = t$.

This shows that for $t \geq 0$ the numbers $S_k^n(t)$ count k-parameter words of length n over a t-element alphabet.

Proof. The assertion is certainly true for $k = 0$, with respect to $k+1$ it follows inductively from the recursions $S_{k+1}^{n+1}(t) = S_k^n(t) + (k+1+t) \cdot S_{k+1}^n(t)$, resp.,

$$\left|[A]\binom{n+1}{k+1}\right| = \left|[A]\binom{n}{k}\right| + (k+1+t) \cdot \left|[A]\binom{n}{k+1}\right|. \quad \square$$

Observe that for $A = \emptyset$ parameter words $f \in [\emptyset]\binom{n}{k}$ are simply surjections $f : \{0, \ldots, n-1\} \to \{\lambda_0, \ldots, \lambda_{k-1}\}$ satisfying $\min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_j)$ for $i < j < k$. They represent equivalent relations on $\{0, \ldots, n-1\}$ with precisely k equivalence classes, yielding the well-known interpretation of the Stirling numbers of the second kind.

For $A = \{0, 1\}$, so $|A| = t = 2$, parameter words $f \in [\{0,1\}]\binom{n}{k}$ represent $\mathcal{P}(k)$-sublattices in $\mathcal{P}(n)$, where $\mathcal{P}(k)$ is the Boolean lattice of subsets of a k-element set.

Consider next the noncentral Bellnumbers

$$B_n(t) = \sum_{k \geq 0} S_k^n(t).$$

Generalizing the wellknown exponential generating function for the Bellnumbers one obtains

$$\sum_{n \geq 0} \frac{B_n(t)}{n!} \cdot z^n = e^{t \cdot z + e^z - 1}.$$

Moreover, each $B_n(t)$ can be written as an alternating sum of ordinary

Bellnumbers, i.e.

$$B_n(t)=\sum_{i=1}^{t}(-1)^{t-i}\cdot a_i(t)\cdot B_{n+i}$$

for certain $a_i(t)\geqslant 0$, where $B_n=B_n(0)$ counts the number of equivalence relations of an n-element set. E.g.

$$B_n(1)=B_{n+1}$$
$$B_n(2)=B_{n+2}-B_{n+1}$$
$$B_n(3)=B_{n+3}-3\cdot B_{n+2}+2\cdot B_{n+1}$$
$$B_n(4)=B_{n+4}-6\cdot B_{n+3}+11\cdot B_{n+2}-6\cdot B_{n+1}$$

In fact, Thumser [11] showed that

$$B_n(k)=\sum_{i=1}^{k}s_i^k\cdot B_{n+i},$$

where the s_i are the Stirling numbers of the first kind.

3. Two observations about $S_k^n(-t)$

In this section t denotes a non*positive* integer. The following three tables show some values of $S_k^n(t)$ for $t=-1,-2,-3$.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	
0	1	0										
1	−1	1	0									
2	1	−1	1	0								
3	−1	1	0	1	0							
4	1	−1	1	2	1	0						
5	−1	1	0	5	5	1	0					
6	1	−1	1	10	20	9	1	0				
7	−1	1	0	21	70	56	14	1	0			
8	1	−1	1	42	231	294	126	20	1	0		
9	−1	1	0	85	735	1 407	924	246	27	1	0	
10	1	−1	1	170	2 290	6 363	6 027	2 400	435	35	1	$t=-1$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	
0	1	0										
1	−2	1	0									
2	+4	−3	1	0								
3	−8	7	−3	1	0							
4	+16	−15	7	−2	1	0						
5	−32	31	−15	5	0	1	0					
6	+64	−63	31	−10	5	3	1	0				
7	−128	127	−63	21	0	14	7	1	0			
8	+256	−255	127	−42	21	42	42	12	1	0		
9	−512	511	−255	85	0	147	210	102	18	1	0	
10	+1 024	−1 023	511	−170	85	441	987	720	210	25	1	$t=-2$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	
0	1	0										
1	−3	1	0									
2	9	−5	1	0								
3	−27	19	−6	1	0							
4	81	−65	25	−6	1	0						
5	−243	211	−90	25	−5	1	0					
6	729	−665	301	−90	20	−3	1	0				
7	−2 187	2059	−966	301	−70	14	0	1	0			
8	6 561	−6 305	3 025	−966	231	−42	14	4	1	0		
9	−19 683	19 171	−9 330	3 025	−735	147	0	30	9	1	0	
10	59 049	−58 025	28 501	−9 330	2 290	−441	147	120	75	15	1	$t=-3$

One easily observes that each row eventually contains nonnegative entries only. Moreover,

$$S_{2t}^{2n+1}(-t)=0 \quad \text{and} \quad S_{2t+2}^{2n+2}(-t-1)=S_{2t+1}^{2n+1}(-t).$$

This is true in general:

Lemma 1. $S_{2t}^{2n+1}(-t)=0.$

Proof. Recall that $S_{2t}^{2n+1}(-t)=\sum_{a=-t}^{t} a^{2n+1}\cdot\prod_{b=-t,b\neq a}^{t}(a-b)^{-1}$, i.e.

$$S_{2t}^{2n+1}(-t)=\sum_{a=-t}^{-1} a^{2n+1}\cdot\prod_{b=-t,b\neq a}^{t}(a-b)^{-1} + \sum_{a=1}^{t} a^{2n+1}\cdot\prod_{b=-1,b\neq a}^{t}(a-b)^{-1},$$

however,

$$\begin{aligned}\sum_{a=-t}^{-1} a^{2n+1}\cdot\prod_{b=-t,b\neq a}^{t}(a-b)^{-1} &= -\sum_{a=1}^{t} a^{2n+1}\cdot\prod_{b=-t,b\neq -a}^{t}(-a-b)^{-1}\\ &= -\sum_{a=1}^{t} a^{2n+1}\cdot\prod_{b=-t,b\neq a}^{t}(a-b)^{-1}. \quad \square\end{aligned}$$

Lemma 2. $S_{2t+2}^{2n+2}(-t-1)=S_{2t+1}^{2n+1}(-t).$

Proof.

$$\begin{aligned}S_{2t+2}^{2n+2}(-t-1) &= S_{2t+1}^{2n+1}(-t-1)+(t+1)\cdot S_{2t+2}^{2n+1}(-t-1)\\ &= S_{2t+1}^{2n+1}(-t-1),\end{aligned}$$

hence it suffices to show that

$$S_{2t+1}^{2n+1}(-t-1)=S_{2t+1}^{2n+1}(-t).$$

But

$$\begin{aligned} S_{2t+1}^{2n+1}(-t-1) &= \sum_{a=-t-1}^{t} a^{2n+1} \cdot \prod_{b=-t-1, b\neq a}^{t} (a-b)^{-1} \\ &= \sum_{a=-t}^{t+1} (-a)^{2n+1} \prod_{b=-t, b\neq a}^{t+1} -(a-b)^{-1} \\ &= S_{2t+1}^{2n+1}(-t). \quad \square \end{aligned}$$

4. A combinatorial interpretation of $S_k^n(t)$ for $t<0$

Recall that for $t \geqslant 0$ the numbers $S_k^n(t)$ count k-parameter words of length n over t-element alphabet. Analogously, also for $t \leqslant 0$ the numbers $S_k^n(t)$ count certain words of length n. However, as the tables already indicate, such an interpretation only makes sense for $2t \leqslant k \leqslant n$.

Notation. By

$$\frac{\lambda_i \lambda_{i+1} \cdots \lambda_{2i}}{2 \cdot l_i}$$

we denote a typical word of length $2 \cdot l_i$ over alphabet $\{\lambda_i, \ldots, \lambda_{2i}\}$. By abuse of language we denote by this symbol the set of all such words,

$$\frac{\lambda_i \lambda_{i+1} \cdots \lambda_{2i}}{2 \cdot l_i} = \{\lambda_1, \ldots, \lambda_{2i}\}^{2 \cdot l_i}.$$

Definition. $\mathcal{M}_0^n(t)$ is the set of all words of length $2t+n$ over alphabet $\{\lambda_0, \ldots, \lambda_{2t-1}\}$ which can be written as

$$\left(\lambda_0 \frac{\lambda_0}{2 \cdot l_0} \lambda_1\right)\left(\lambda_2 \frac{\lambda_1 \lambda_2}{2 \cdot l_1} \lambda_3\right) \cdots \left(\lambda_{2i} \frac{\lambda_i \lambda_{i+1} \cdots \lambda_{2i}}{2 \cdot l_i} \lambda_{2i+1}\right) \cdots$$
$$\cdots \left(\lambda_{2t-2} \frac{\lambda_{t-1} \lambda_t \cdots \lambda_{2t-2}}{2 \cdot l_{t-1}} \lambda_{2t-1}\right),$$

where the l_i's are nonnegative integers such that $2 \cdot (l_0 + \cdots + l_{t-1}) = n$. A word in $\mathcal{M}_0^n(t)$ which is characterized by $(l_0, \ldots, l_{t-1})$ is the concatenation of words $\bar{a}_i$, where $\bar{a}_i$ is of length $2(l_i+1)$, it starts with λ_{2i}, then comes a sequence of length $2l_i$ of symbols in $\{\lambda_i, \lambda_{i+1}, \ldots, \lambda_{2i}\}$ and it finishes with λ_{2i+1}.

So, by definition, $\mathcal{M}_0^{2n+1}(t) = \emptyset$ and

$$|\mathcal{M}_0^{2n}(t)| = \sum_{0=v_0 \leqslant v_1 \leqslant \cdots \leqslant v_t = n} \prod_{a=1}^{t} a^{2 \cdot (v_a - v_{a-1})}$$

Our claim will be that $S_{2t}^{2t+2n}(-t) = |\mathcal{M}_0^{2n}(t)|$. But before we can prove this, sets $\mathcal{M}_k^n(t)$ have to be defined for general k.

Notation. For nonnegative integers t, k and m we denote by $[t, k, m]$ the set of words $f:\{0,\dots,m-1\}\to\{\lambda_t, \lambda_{t+1},\dots,\lambda_{2t},\lambda_{2t+1},\dots,\lambda_{2t+k-1}\}$ satisfying

(1) $\quad f^{-1}(\lambda_{2t+i})\neq\emptyset$ for all $1<i<k$ and

(2) $\quad \min f^{-1}(\lambda_{2t+i})<\min f^{-1}(\lambda_{2t+j})$ for all $1<i<j<k$.

By abuse of language, $[t, k, m]$ also denotes a typical element of $[t, k, m]$.

Remark. Observe the formal analogy between parameter words $f\in[A]\binom{m}{k-1}$ for a $(t+1)$-element alphabet A and words $f\in[t,k,m]$. In particular,

$$\|[t,k,m]\| = \left|[A]\binom{m}{k-1}\right| = S^m_{k-1}(t+1).$$

Definition. For $k>0$ let $\mathcal{M}^n_k(t)$ be the set of all words of length $2t+n$ over alphabet $\{\lambda_0,\lambda_1,\dots,\lambda_{2t+k-1}\}$ which can be written as

$$\Big(\lambda_0\frac{\lambda_0}{2\cdot l_0}\lambda_1\Big)\Big(\lambda_2\frac{\lambda_1\lambda_2}{2\cdot l_1}\lambda_3\Big)\cdots\Big(\lambda_{2i}\frac{\lambda_i\lambda_{i+1}\cdots\lambda_{2i}}{2\cdot l_i}\lambda_{2i+1}\Big)$$
$$\cdots\Big(\lambda_{2t-2}\frac{\lambda_{t-1}\lambda_t\cdots\lambda_{2t-2}}{2l_{t-1}}\lambda_{2t-1}\Big)\lambda_{2t}[t,k,n-1-2\cdot(l_0+\cdots+l_{t-1})], \quad (**)$$

where the l_i's are nonnegative integers such that $2\cdot(l_0+\cdots+l_{t-1})<n$.

Let us summarize a few basic observations about these sets $\mathcal{M}^n_k(t)$:

Lemma 3. *Let t and $k\le n$ be nonnegative integers. Then*

(a) $\quad |\mathcal{M}^{2n+1}_0(t)|=0.$

(b) $\quad |\mathcal{M}^{n+1}_{k+1}(t)| = |\mathcal{M}^n_k(t)| + (t+k+1)\cdot|\mathcal{M}^n_{k+1}(t)|,$

(c) $\quad |\mathcal{M}^{2n}_0(t+1)| = |\mathcal{M}^{2n+1}_1(t)|,$

(d) $\quad |\mathcal{M}^{2n}_0(t)| = \displaystyle\sum_{0\le v_0\le v_1\le\cdots\le v_t=n}\ \prod_{a=1}^{t} a^{2\cdot(v_a-v_{a-1})},$

(e) $\quad |\mathcal{M}^n_{k+1}(t)| = \displaystyle\sum_{0\le l\le n/2} |\mathcal{M}^{2l}_0(t)|\cdot S^{n-1-2l}_k(t+1).$

Proof. (a) is obvious from the definition.

ad (b). Consider $f=(f_0,\dots,f_n)\in\mathcal{M}^{n+1}_{k+1}(t)$. If $(f_0,\dots,f_{n-1})\in\mathcal{M}^n_k(t)$, then necessarily $f_n=\lambda_k$ and f is determined uniquely from $(f_0,\dots,f_{n-1})$. If $(f_0,\dots,f_{n-1})\in\mathcal{M}^n_{k+1}(t)$, then there are $(t+k+1)$ many possibilities for f_n, viz. $f_n\in\{\lambda_t,\dots,\lambda_{t+\cdot}\}$. As $(f_0,\dots,f_{n-1})\in\mathcal{M}^n_k(t)\cup\mathcal{M}^n_{k+1}(t)$ assertion (b) follows.

ad (c). Consider a typical word from $\mathcal{M}_0^{2n}(t+1)$, viz.

$$\left(\lambda_0 \frac{\lambda_0}{2\cdot l_0}\lambda_1\right)\cdots\lambda_{2t}\frac{\lambda_t\lambda_{t+1}\cdots\lambda_{2t}}{2\cdot l_t}\lambda_{2t+1},$$

deleting the last entry, i.e. λ_{2t+1}, yields a word in $\mathcal{M}_1^{2n+1}(t)$ and, as $2n$ is even, this is a bijective correspondence.

ad (d). Obvious from (c).

ad (e). By definition and as $\|[t,k,m]\| = S_{k-1}^m(t+1)$ it follows that

$$|\mathcal{M}_{k+1}^n(t)| = \sum_{0\le v_0\le\cdots\le v_t<n/2}\left(\prod_{a=1}^t a^{2(v_a-v_{a-1})}\right)\cdot S_t^{n-1-2v}(t+1)$$

$$= \sum_{0\le l<n/2} |\mathcal{M}_0^{2l}(t)|\cdot S_k^{n-1-2l}(t+1). \quad \square$$

Now we formulate and prove the principal result of this paper:

Theorem. $S_{2t+k}^{2t+n}(-t) = |\mathcal{M}_k^n(t)|$, *for nonnegative integers t and* $k\le n$.

This shows that for $k=0$ $S_{2t+k}^{2t+n}(t)$ counts the number of words of length $2t+n$ over alphabet $\{\lambda_0,\ldots,\lambda_{2t+k-1}\}$ of shape (*) if $k=0$ and for $k>0$ those of shape (**).

Remark. A similar description for $t=1$ occurs in [12].

Proof. We proceed by induction on t. For $t=0$ one easily observes that $\mathcal{M}_k^n(0) = [\emptyset]\binom{n}{k}$ and nothing is to show. So assume that $S_{2t+k}^{2t+n}(-t) = |\mathcal{M}_k^n(t)|$ for all k and n.

We first show that $S_{1(t+1)}^{2(t+1)+n}(-t-1) = |\mathcal{M}_0^n(t+1)|$ for all nonnegative integers n. By Lemma 1, resp., Lemma 3(a) this is obvious for odd n. by induction it follows that $|\mathcal{M}_1^{2n+1}(t)| = |S_{2t+1}^{2t+2n+1}(-t)|$. Hence, together with Lemma 2 and Lemma 3(c) we obtain

$$\begin{aligned}|\mathcal{M}_0^{2n}(t+1)| &= |\mathcal{M}_1^{2n+1}(t)|\\ &= |S_{2t+1}^{2t+2n+1}(-t)|\\ &= S_{2(t+1)}^{2(t+1)+2n}(-t-1).\end{aligned}$$

Obviously $S_{2(t+1)+n}^{2(t+1)+n}(-t-1) = 1 = |\mathcal{M}_n^n(t+1)|$. Hence we may use the recursions of Lemma 3(b) and the corresponding recursions for $S_k^n(-t-1)$ in order to complete the proof. $\square$

Finally, we mention the following corollary (cf. Lemma 3(e)):

Corollary. $S_{2t+k}^{2t+n}(-t) = \sum_{0\le l\le n/2} S_{2t}^{2t+2l}(-t)\cdot S_k^{n-1-2l}(1+t)$ *for nonnegative integers t and* $k\le n$.

References

[1] G. Bach, Über eine Verallgemeinerung der Differenzengleichung der Stirling'schen Zahlen 2. Art und einige damit zusammenhängenden Fragen, Zeitschrift für die reine und angewandte Mathematik 233 (1968) 213–220.
[2] L. Carlitz, Weighted Stirling numbers of the first and second kind I, Fibonacci Quarterly 18 (1980) 147–162.
[3] L. Comtet, Nombre de Stirling generaux et functions symetrique, C.R. Acad. Sci. Paris 275 (1972) Series A, 747–750.
[4] W. Deuber and B. Voigt, Partitionseigenschaften endlicher affiner under projektiver Räume, Europ. J. Combinat. 3 (1982) 329–340.
[5] R.L. Graham and B.L. Rothschild, Ramsey's theorem for n-parameter sets, Trans. Amer. Math. Soc. 159 (1971) 257–292.
[6] R.L. Graham, B.L. Rothschild and J.H. Spencer, Ramsey Theory (New York 1980: John Wiley).
[7] Ch. Jordan, Calculus of Finite Differences, Reprint 1965 (Chelsea Publ. Comp. New York).
[8] M. Koutras, Noncentral Stirling numbers and some applications, Discrete Math. 42 (1982) 73–89.
[9] K. Leeb, Vorlesungen über Pascaltheorie (Erlangen, 1973).
[10] H.J. Prömel and B. Voigt, Graham–Rothschild parameter sets, to appear in The Mathematics of Ramsey Theory (ed. J. Nesetril, V. Rödl).
[11] W. Thumser, in preparation.
[12] B. Voigt, A common generalization of binomial coefficients, Stirling numbers and Gonssions coefficients, Proceedings of the 11th Winter School, Suppl. ai Rendiconti del Circolo Matematico Palermo, Serie II, 3 (1984) 339–359.